Sydney Technical High School

Student Name ______________________________

Teacher's Name: ______________________________

Extension 2 Mathematics

TRIAL HSC

August 2021

General Instructions

- Reading time – 10 minutes
- Working time – 180 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt questions 11-16
- Allow about 2 hours and 45 minutes for this section

Section I 10 Marks

Attempt Questions 1-10

Allow approximately 15 minutes for this section.

Use the Multiple Choice answer sheet for questions 1-10.

1. Which vector is perpendicular to

$$\begin{pmatrix} 6 \\ -4 \\ 8 \end{pmatrix}?$$

A) $\begin{pmatrix} 12 \\ -8 \\ 16 \end{pmatrix}$ B) $\begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix}$ C) $\begin{pmatrix} 6 \\ 12 \\ 2 \end{pmatrix}$ D) $\begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix}$

2. A particle is describing SHM in a straight line with an amplitude of 2 metres. Its speed is 3m/s when the particle is 1 metre from the centre of motion.

What is the period of the motion?

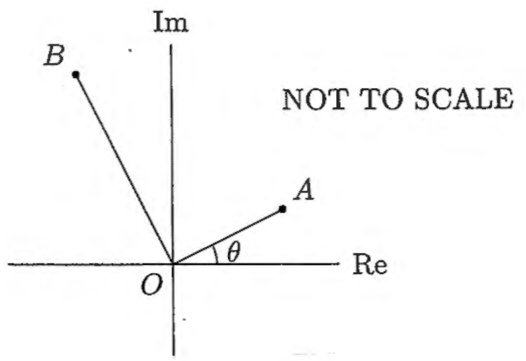
A) $\frac{\pi\sqrt{3}}{2}$ B) $\frac{2\pi\sqrt{3}}{3}$ C) $\pi\sqrt{3}$ D) $\frac{2\pi\sqrt{2}}{3}$

3. Consider the following statement for $n\epsilon\mathbb{Z}$.

If $n^2 + 4n$ is odd, then n is odd. Which of the following statements is the contrapositive of this statement for $n\epsilon\mathbb{Z}$?

A) If n is even, then $n^2 + 4n$ is even
B) If $n^2 + 4n$ is even then n is even
C) If n is odd then $n^2 + 4n$ is odd
D) If $n^2 + 4n$ is odd then n is even

4. The points A and B in the diagram represent the complex numbers z_1 and z_2 respectively, where $|z_1| = 1$ and arg $(z_1)=\theta$ and $z_2= \sqrt{3}iz_1$



Which of the following represents $z_2 - z_1$?

A) $2e^{i(\frac{2\pi}{3}+\theta)}$ B) $3e^{i(\frac{2\pi}{3}+\theta)}$ C) $2e^{i(\frac{2\pi}{3}-\theta)}$ D) $3e^{i(\frac{2\pi}{3}-\theta)}$

5. Which of the following is an expression for $\int_0^{\frac{\pi}{2}} \frac{1}{1+cosx}dx$ after using the substitution $t = \tan\frac{x}{2}$?

A) $\int_0^1 \frac{1}{1+2t}dt$ B) $\int_0^1 1+t^2\,dt$ C) $\int_0^1 dt$ D) $\int_0^1 \frac{2}{(1+t)^2}dt$

6. Which of the following is false?

A) $\int_{-3}^{3} x^3\, e^{-x^2} dx = 0$

B) $\int_{-4}^{4} \frac{x^2}{x^2+4}dx = 2\int_0^4 \frac{x^2}{x^2+4}dx$

C) $\int_0^{\pi} sin^4\theta d\theta > \int_0^{\pi} sin4\theta d\theta$

D) $\int_0^1 x^4\, dx < \int_0^1 x^5\, dx$

7. The points A, B and C are collinear where

$$\overrightarrow{OA} = \underset{\sim}{i} + \underset{\sim}{j}, \quad \overrightarrow{OB} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}, \quad \overrightarrow{OC} = 3\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}.$$

What are the values of a and b?

A) $a = 3,\ b = 2$ B) $a = -3,\ b = 2$

C) $a = 3,\ b = -2$ D) $a = -3,\ b = -2$

8. If z is any complex number satisfying $|z-2|=1$, then which of the following is correct?

A) $\arg(z-2)=2\arg(z-1)$
B) $2argz=\frac{2}{3}\arg(z^2+2z)$
C) $argz=2\arg(z-2)$
D) $\arg(z+2)=\arg(z-2)$

9. Let $\underset{\sim}{u}=\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\underset{\sim}{v}=\begin{pmatrix}1\\2\\2\end{pmatrix}$.

The angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is

A) 30° B) 22.5° C) 0° D) 45°

10. Given that $w^7=1$ and w is a complex number, what is the value of

$1+w+w^2+w^3+w^4+w^5+w^6+w^7$?

A) 1 B) w C) $-w$ D) 0

Section II 90 Marks

Attempt Questions 11-16.

Allow approximately 2 hours and 45 minutes for this section.

Your responses should include relevant mathematical reasoning and/or calculations.

Begin each question on a NEW page.

Question 11 **(15 marks)** **Marks**

a) Let $z = 1 + \sqrt{3}i.$ Find the complex numbers for

(i) $z\bar{z}$ 2

(ii) z^2 2

b) Express $-2 + 2\sqrt{3}i$ in the form $re^{i\theta}$ 2

c) Find

(i) $\displaystyle\int \frac{dx}{x(lnx)}$ 2

(ii) $\displaystyle\int \frac{1}{x^2 + 6x + 13}dx$ 2

d) (i) Use Euler's formula $e^{i\theta} = cos\theta + isin\theta$ to find a similar simplified result for $e^{-i\theta}$ 1

(ii) Hence show $sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ 1

(iii) Use the result from part (ii) to show that

$$sin^3\theta = \frac{3}{4}sin\theta - \frac{1}{4}\ sin3\theta$$ 3

End of Question 11

Question 12 **(15 marks)** **Begin a NEW page** **Marks**

a) The velocity v m/s of a particle moving in simple harmonic motion along the x axis is given by $v^2 = 8 - 2x - x^2$.

(i) Between which two points is the particle oscillating? 2

(ii) Find the acceleration of the particle in terms of x 1

(iii) Find the period of the motion 1

(iv) What is the time taken by the particle to travel the first 60m of its motion (leave in exact form) 1

b) Given that *a, b* and *c* are positive real numbers, prove that

$$(a+b+c)^2 \geq 3(ab+ac+bc)$$ 3

c) If $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$, find the unit vector(s) perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ 2

d) The points *P, Q* and *R* have position vectors $\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{i} + 2\underset{\sim}{j}$ and $2\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$ respectively.

(i) Find the vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$ 1

(ii) Find the magnitude of $\angle PQR$ correct to the nearest degree 2

e) (i) Find the vector equation of line vector $\underset{\sim}{r}$ given it passes through $(1, 3, -2)$ and $(2, -1, 2)$ 1

(ii) Determine if $\underset{\sim}{r}$ passes through $(4, -9, 10)$ 1

End of Question 12

Question 13 **(15 marks)** **Begin a NEW page** **Marks**

a) Find $\int \frac{dx}{x\sqrt{x^2 - 4}}$ using the substitution $u = \sqrt{x^2 - 4}$ 2

b) Evaluate $\int x^2 sinxdx$ 2

c) Given that for $k > 0, \frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$ use Mathematical Induction to prove that for all integers $n \geq 2$, 3

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

d) If $P = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $R = 9\underset{\sim}{i} + 3\underset{\sim}{j} + 8\underset{\sim}{k}$ find the point Q on $\overrightarrow{PR}$ such that $PQ : QR = 2 : 3$ 3

e) The rise and fall of tides can be approximated to simple harmonic motion. At 11am the depth of water in a tidal lagoon is lowest at 1m. The following high tide is at 5:21pm with a depth of 5m.

(i) Find the period of this motion in minutes 1

(ii) Express the motion using a cosine function 2

(iii) Calculate between which times a yacht could safely cross the lagoon if a minimum depth of 2m is required 2

End of Question 13

Question 14 **(15 marks)** **Begin a NEW page** **Marks**

a) Prove the following statement using a proof by contradiction. 3

"For each irrational number s, the number $2s+1$ is also irrational"

b) (i) If $I_n = \int_0^{\frac{\pi}{4}} sec^n \theta d\theta$ show that 3

$$I_n = \frac{1}{n-1}\left[(n-2)I_{n-2} + (\sqrt{2})^{n-2}\right]$$

for $n \geq 2$

(ii) Evaluate $\int_0^{\frac{\pi}{4}} sec^4 \theta d\theta$ 2

c) (i) Find real numbers a, b and c such that 2

$$\frac{1}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}$$

(ii) Hence find $\int \frac{1}{x^2(x-1)} dx$ in simplest form. 2

d) Find a point of intersection of the line with parametric equation 3

$r = \underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k} + t(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$ and the sphere with equation

$(x-1)^2 + (y-3)^2 + (z+4)^2 = 81$

End of Question 14

Question 15 **(15 marks)** **Begin a NEW page** **Marks**

a) The point $P(x, y)$ representing the complex number z moves in the Argand diagram so that $| z \text{-}6 |=| z +2i |$.

(i) Show that the path P traces out, has equation $3x + y - 8 = 0$ 2

(ii) Find the minimum value of $|z|$ as P moves on this path 2

b) The polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has zeros $a + bi$ and $a + 2bi$ 3

where a and $b \in \mathbb{R}$ and $b \neq 0$.

By evaluating a and b, find all the zeros of $P(x)$.

c) A particle moves along the $x -$ axis, starting at $x = 0.1$ at time $t = 0$.

The velocity of the particle is described by

$$v = \sqrt{2x}\, e^{-x^2} \quad x \geq 0.1$$

where x is the displacement of the particle from the origin.

(i) Show that the acceleration of the particle is given by 2

$$a = e^{-2x^2}(1\text{-}4x^2)$$

(ii) Hence find the fastest speed attained by the particle. 2

(iii) Show that T, the time taken to travel from $x = 1$ to $x = 2$ can be expressed as 2

$$T = \int_1^2 \frac{1}{\sqrt{2x}}\, e^{x^2} dx$$

(iv) Use the trapezoidal rule with three function values to give an approximate value of T correct to the nearest whole number. 2

End of Question 15

Question 16 (15 marks) **Begin a NEW page** **Marks**

a) Given $\overrightarrow{AP} = \begin{bmatrix} -1 \\ 6 \\ -1 \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} -2 \\ 5 \\ -7 \end{bmatrix}$

(i) Find $proj_{\underset{\sim}{b}}\, \underset{\sim}{p}$ where $\overrightarrow{AP} = \underset{\sim}{p}$ and $\overrightarrow{AB} = \underset{\sim}{b}$ 2

(ii) Find the perpendicular distance d from P to the line AB. 2

b) (i) Using a suitable substitution, show that 2

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

(ii) A function $f(x)$ has the property that $f(x) + f(a-x) = f(a)$ 2

Using part (i), or otherwise, show that

$$\int_0^a f(x)dx = \frac{a}{2} f(a)$$

c) (i) Given $f(t) = \sin(a + nt)\, sinb - sinasin(b - nt)$ 2

where a, b and n are constants with $a > 0, b > 0$ and $a + b < \pi$ and $n \neq 0$,

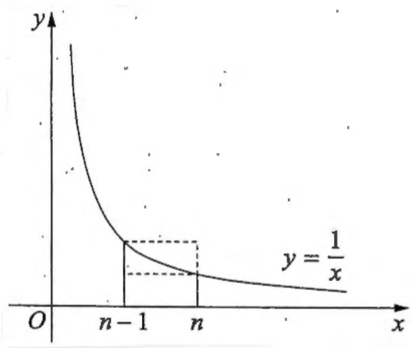
show that $f(t) = \sin(a + b)\, sinnt$

(ii) Find all values of t for which 3

$$\frac{\sin\,(a+nt)}{\sin\,(b-nt)} = \frac{sina}{sinb}$$

Question 16 continues………

d) 2



Let n be a positive integer greater than 1. The area of the region under the curve $y = \frac{1}{x}$ from $x = n - 1$ to $x = n$ is between the areas of two rectangles, as shown in the diagram.

Show that $e^{\frac{-n}{n-1}} < (1 - \frac{1}{n})^{\mathrm{n}} < e^{-1}$

End of Exam

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2021 Ext. 2 Trial HSC solutions

1. only one where dot product $= 0$ is D.

2. $v^2 = n^2(a^2 - x^2)$

$3^2 = n^2(2^2 - 1^2)$

$9 = 3n^2$

$n^2 = 3 \quad n = \sqrt{3}$

$T = \frac{2\pi}{\sqrt{3}} = \frac{2\sqrt{3}\pi}{3}$

B

3. A

4. $z_2 - z_1$

$= \sqrt{3} i z_1 - z_1$

$= \text{cis}\,\theta(-1 + \sqrt{3} i)$

$= \text{cis}\,\theta \times 2\,\text{cis}\left(-\frac{\pi}{3}\right)$

$= 2\,\text{cis}\left(\theta - \frac{2\pi}{3}\right)$

$= 2e^{i(\theta - \frac{\pi}{3})}$

A

5. $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\, dx$

$t = \tan\frac{x}{2}$

$x = 0 \Rightarrow t = 0$

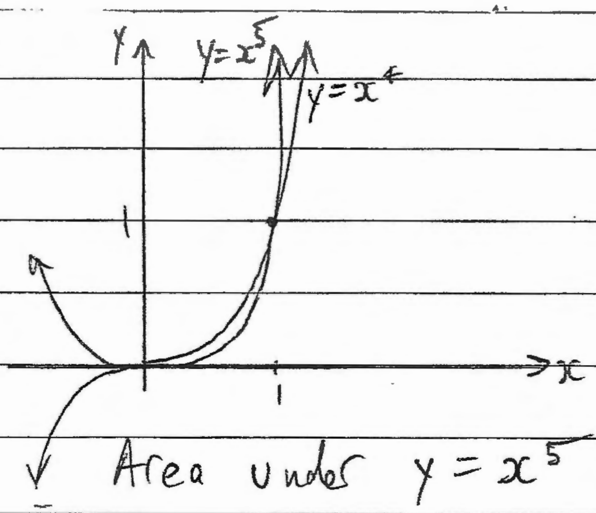
$x = \frac{\pi}{2} \Rightarrow t = 1$

$\int_0^1 \frac{1}{1 + \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$

$\int_0^1 \frac{2dt}{1+\cancel{t^2}+1-\cancel{t^2}} = \int_0^1 dt$

C

6. D



Area under $y = x^5$ is lower between $x = 0$ and $x = 1$.

7. $\frac{2}{1} = \frac{a-1}{-2} = \frac{b}{1}$

$\therefore a = -3,\ b = 2$

B

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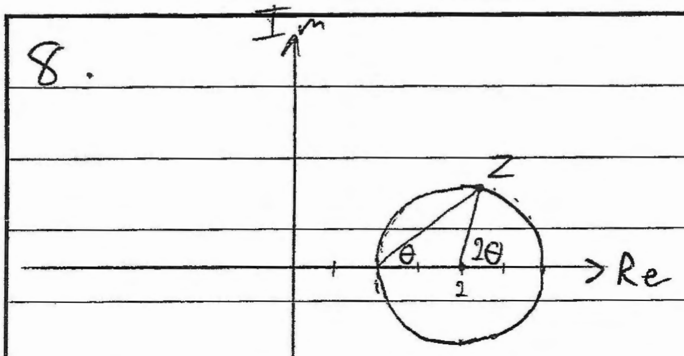
8.



A is true as exterior angle equals twice interior opp. angle in isosceles Δ

A

9. $\cos\theta = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}| \times |\underset{\sim}{v}|}$

$= \dfrac{1\times 1 + 1\times 2 + 0\times 2}{\sqrt{1^2+1^2+0^2} \times \sqrt{1^2+2^2+2^2}}$

$= \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\therefore \theta = 45^\circ$

D

10. $\omega^7 - 1 = 0$

$(\omega - 1)(\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$

If ω is complex then 2nd bracket $= 0$.

but $\omega^7 = 1$

$\therefore \underbrace{1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6}_{0} \underset{+ 1}{+ \omega^7}$

$= 1$

A

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1(a) $z = 1+\sqrt{3}i$

(i) $z\bar{z} = (1+\sqrt{3}i)(1-\sqrt{3}i)$ 1

$= 1^2 + (\sqrt{3})^2$

$= 4$ 2

(ii) $z^2 = (1+\sqrt{3}i)^2$

$= 1+2\sqrt{3}i + 3i^2$ 1

$= -2+2\sqrt{3}i$ 2

b). $-2+2\sqrt{3}i$

$r = \sqrt{(-2)^2+(2\sqrt{3})^2}$

$= \sqrt{4+12}$

$= 4$ 1

$\arg = \tan^{-1}\frac{2\sqrt{3}}{-2}$

$= \tan^{-1}(-\sqrt{3})$

working angle is $\frac{\pi}{3}$

2nd quadrant $\therefore \frac{2\pi}{3}$ 1

so $-2+2\sqrt{3}i = 4e^{\frac{2\pi i}{3}} = 4e^{i(\frac{2\pi}{3})}$ 2

c) (i) $\int \frac{dx}{x(\ln x)}$

Let $u = \ln x$

$\therefore \frac{du}{dx} = \frac{1}{x}$

$\therefore du = \frac{dx}{x}$

$\int \frac{du}{u}$ 1

$= \log_e u$

$= \log_e|(\log_e x)| + c$ 2

(ii) $\int \frac{1}{x^2+6x+13}\,dx$

$\int \frac{1}{(x+3)^2+2^2}\,dx$ 1

$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$

2

d(i) $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$e^{-i\theta} = \cos\theta - i\sin\theta$

1

(ii) $e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)$

$= 2i\sin\theta$

$\therefore \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

1

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c(iii) $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

$\sin^3\theta = \frac{1}{8i^3}(e^{i\theta} - e^{-i\theta})^3$ (1)

$= \frac{-1}{8i}(e^{3i\theta} - 3e^{2i\theta} \times e^{-i\theta} + 3e^{i\theta}e^{-2i\theta} - e^{-3i\theta})$

$= \frac{-1}{8i}(e^{3i\theta} - e^{-3i\theta} - 3(e^{i\theta} - e^{-i\theta}))$ (2)

$= \frac{-1}{8i}(2i\sin 3\theta - 3 \times 2i\sin\theta)$

$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta)$ (3) as required.

12a)(i) $v^2 = 8 - 2x - x^2$

Let $v = 0$

$0 = x^2 + 2x - 8$ (1)

$0 = (x-2)(x+4)$

$x = 2$ and -4 (2)

oscillates between these.

(ii) $\frac{1}{2}v^2 = 4 - x - \frac{x^2}{2}$

$\frac{d}{dx}(\frac{1}{2}v^2) = -1 - \frac{2x}{2}$

$a = -(x+1)$ (1)

(iii) Period $= \frac{2\pi}{n}$

$n=1$ from above

$\therefore$ Period is 2π

(1)

(iv) Travels 12m in one period $\therefore$ 5 periods

$= 5 \times 2\pi = 10\pi$ seconds (1)

b) $(a-b)^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2ab$

$(b-c)^2 \geq 0 \Rightarrow b^2 + c^2 \geq 2bc$

$(a-c)^2 \geq 0 \Rightarrow a^2 + c^2 \geq 2ac$ (1)

$2a^2 + 2b^2 + 2c^2 \geq 2(ab + bc + ac)$

$a^2 + b^2 + c^2 \geq ab + bc + ac$ (2)

$(a+b+c)^2 - 2ab - 2bc - 2ac \geq ab + bc + ac$

$\therefore (a+b+c)^2 \geq 3ab + 3bc + 3ac$

(3)

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12c) Let the unit vector be $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = 0$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0$ gives

1 $3a + 2b + 2c = 0$ ①

$a - b - c = 0$ ②

① + 2 × ②

$5a = 0$

$\therefore a = 0 \quad \therefore b = -c$

unit vector means

$a^2 + b^2 + c^2 = 1$

$\therefore 2b^2 = 1$

$b = \pm\frac{1}{\sqrt{2}} \quad c = \mp\frac{1}{\sqrt{2}}$

$\therefore \underset{\sim}{v} = \pm\left(\frac{1}{\sqrt{2}}\underset{\sim}{j} - \frac{1}{\sqrt{2}}k\right)$ 2

d)(i) $\overrightarrow{QP} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} - (\underset{\sim}{i} + 2\underset{\sim}{j})$

$= -\underset{\sim}{k}$

$\overrightarrow{QR} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - 2k - (\underset{\sim}{i} + 2\underset{\sim}{j})$

$= \underset{\sim}{i} - 5\underset{\sim}{j} - 2\underset{\sim}{k}$

1 for both

(ii) $\overrightarrow{QP} \cdot \overrightarrow{QR} = |QP| \times |QR| \cos\theta$

$2 = 1 \times \sqrt{1^2 + 5^2 + 2^2} \cos\theta$ 1

$\cos\theta = \frac{2}{\sqrt{30}}$

$\theta = 69^\circ$ 2

e)(i) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2-1 \\ -1-3 \\ 2--2 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix}$ 1

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c(ii) Let $\underset{\sim}{r}$ be

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -9 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} \Rightarrow \begin{array}{ll} 4 = 1 + \lambda & \Rightarrow \lambda = 3 \\ -9 = 3 - 4\lambda & \Rightarrow \lambda = 3 \quad ① \\ 10 = -2 + 4\lambda & \Rightarrow \lambda = 3 \end{array}$$

$\therefore$ Line can pass through $(4, -9, 10)$

13 a) $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}$ Let $u = \sqrt{x^2-4}$

$du = \frac{2x}{2\sqrt{x^2-4}}\,dx$

$\displaystyle\int \frac{1}{x} \times \frac{du}{x}$ $\qquad \frac{du}{x} = \frac{dx}{\sqrt{x^2-4}}$

$\displaystyle\int \frac{du}{x^2}$

$\displaystyle\int \frac{du}{u^2+4}$ ① since $u^2 = x^2 - 4$

$= \frac{1}{2}\tan^{-1}\frac{u}{2} + C$

$= \frac{1}{2}\tan^{-1}\frac{\sqrt{x^2-4}}{2} + C$ ②

b) $\displaystyle\int_a^t x^2 \sin x \, dx$

$\displaystyle\int x^2 \frac{d}{dx}(-\cos x)\,dx$

$= -x^2\cos x - \int 2x \times -\cos x \, dx$

$= -x^2\cos x + 2\int x \frac{d}{dx}(\sin x)\,dx$ ①

$= -x^2\cos x + 2\left[x\sin x - \int \sin x \, dx\right]$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$ ②

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c) Show true for $n=2$

$$\frac{1}{1^2} + \frac{1}{2^2} < 2 - \frac{1}{2}$$

$$1 + \frac{1}{4} < 1\frac{1}{2} \checkmark$$

so true for $n=2$

Assume true for $n=k$ where $k>2$

ie: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{k^2} < 2 - \frac{1}{k}$

Now to show true for $n=k+1$

Using assumption (1)

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$< 2 - \frac{1}{k+1} + \left(\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1}\right)$$

$$< 2 - \frac{1}{k+1} \text{ given } \frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$$

$\therefore$ True for $n=k+1$ (2)

Since result is true for $n=2$, it must also be true for $n=2+1=3$, $n=3+1=4$ and by induction for all positive integral values of $n \geq 2$. (3)

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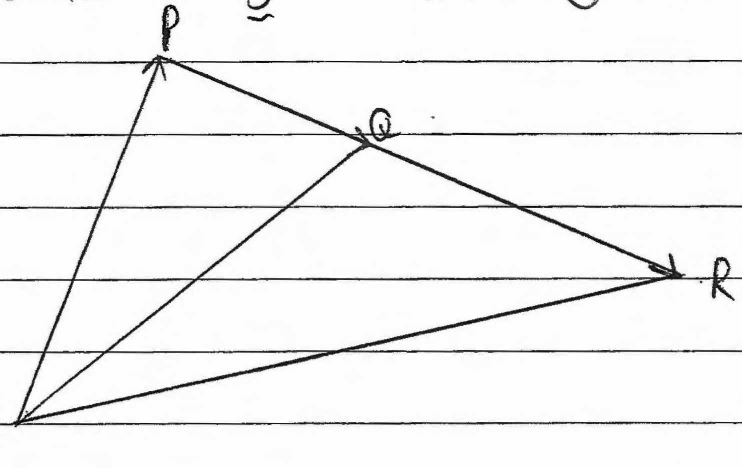
d) $\overrightarrow{PR} = (9\underset{\sim}{i} + 3\underset{\sim}{j} + 8\underset{\sim}{k}) - (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

$= 8\underset{\sim}{i} + 2\underset{\sim}{j} + 7\underset{\sim}{k}$

$\overrightarrow{PQ} = \frac{2}{5}(8\underset{\sim}{i} + 2\underset{\sim}{j} + 7\underset{\sim}{k})$ (1)



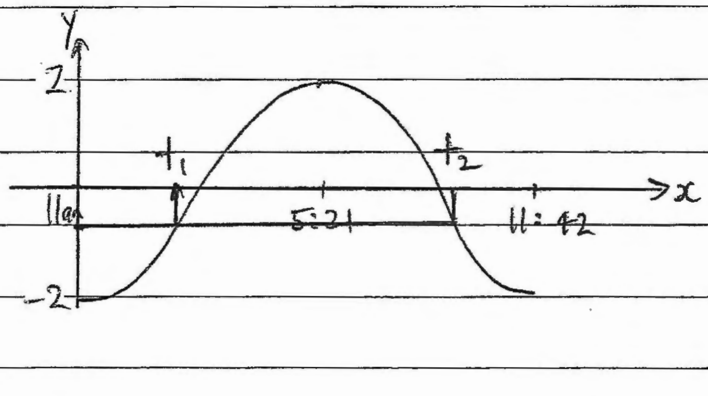
Now $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

$\therefore \overrightarrow{OQ} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} + \frac{2}{5}(8\underset{\sim}{i} + 2\underset{\sim}{j} + 7\underset{\sim}{k})$ (2)

$\overrightarrow{OQ} = \frac{21}{5}\underset{\sim}{i} + \frac{9}{5}\underset{\sim}{j} + \frac{19}{5}\underset{\sim}{k}$

$\therefore Q = \frac{21}{5}\underset{\sim}{i} + \frac{9}{5}\underset{\sim}{j} + \frac{19}{5}\underset{\sim}{k}$ or $\left(\frac{21}{5}, \frac{9}{5}, \frac{19}{5}\right)$ (3)

e)



(i) Period = 12 hrs 42 min

= 762 mins. (1)

(ii) $y = -2\cos nt$ (1)

$T = 762 = \frac{2\pi}{n}$

$\therefore n = \frac{\pi}{381}$

$\therefore y = -2\cos\frac{\pi t}{381}$ (2)

(iii) 1m above low tide $\Rightarrow$ $y = -1$ on our graph

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$$-1 = -2\cos\frac{\pi t}{381} \quad ①$$

$$\frac{1}{2} = \cos\frac{\pi t}{381}$$

$$\therefore \frac{\pi t}{381} = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$t = \frac{381}{3}, \frac{5 \times 381}{3}$$

$$= 127, 635$$

= 2hrs 7min and 10 hrs 35 min after 11am ie: safe between 1:07pm and 9:35pm. ②

14.a) Assume the statement is false that for an irrational number s, there exists a number 2s+1 which is rational

ie: $2s+1 = \frac{a}{b}$ ① where a and b are integers with no common factors.

$$\therefore 2s = \frac{a}{b} - 1$$

$$= \frac{a}{b} - \frac{b}{b}$$

$$2s = \frac{a-b}{b}$$

$$s = \frac{a-b}{2b}$$

$\therefore s = \frac{c}{d}$ ② where c and d are integers since a and b are.

$\therefore$ s is rational

CONTRADICTION ③

$\therefore$ Original statement is true.

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b)(i) $I_n = \int_0^{\frac{\pi}{4}} \sec^n\theta \, d\theta$

$= \int_0^{\frac{\pi}{4}} \sec^2\theta \sec^{n-2}\theta \, d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{d}{d\theta}(\tan\theta) \sec^{n-2}\theta \, d\theta$ ①

$= \left[\tan\theta \sec^{n-2}\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan\theta \times (n-2)\sec^{n-3}\theta \times \sec\theta\tan\theta \, d\theta$

$= (\sqrt{2})^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \tan^2\theta \sec^{n-2}\theta \, d\theta$

$= (\sqrt{2})^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\sec^{n-2}\theta \, d\theta$

$= (\sqrt{2})^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^n\theta \, d\theta + (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2}\theta \, d\theta$

$I_n = (\sqrt{2})^{n-2} - (n-2)I_n + (n-2)I_{n-2}$ ②

$I_n + (n-2)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$

$(n-1)I_n = (n-2)I_{n-2} + (\sqrt{2})^{n-2}$

$I_n = \frac{1}{n-1}\left[(n-2)I_{n-2} + (\sqrt{2})^{n-2}\right]$ ③

(ii) $I_4 = \frac{2}{3}I_2 + \frac{1}{3}(\sqrt{2})^2$ ①

$= \frac{2}{3}(0+1) + \frac{2}{3}$

$I_4 = \frac{4}{3}$ ②

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c)(i) $1 = ax(x-1) + b(x-1) + cx^2$

Let $x=1$ $\therefore 1 = c$ ① for one of these

$x=0$ $\therefore 1 = -b$

Let $x=2$ $\therefore 1 = 2a + b + 4c$

$1 = 2a - 1 + 4$

So $a=-1$, $b=-1$, $c=1$ ②

(ii) $\int \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\, dx$

$= -\log_e|x| + x^{-1} + \log_e|x-1|$ ①

$= \log_e\left|\frac{x-1}{x}\right| + \frac{1}{x} + C$ ②

d) $x = 1 + t$

$y = 3 + 2t$

$z = -4 + 2t$ ①

Sub. these into sphere

$(1+t-1)^2 + (3+2t-3)^2 + (-4+2t+4)^2 = 81$

$t^2 + (2t)^2 + (2t)^2 = 81$

$9t^2 = 81$

$t^2 = 9$

$t = \pm 3$ ②

sub. into ① ② ③

gives $(4, 9, 2)$ or $(-2, -3, -10)$

(one of) ③

15.d)(i) $|z-6| = |z+2i|$

Let $x + iy = z$

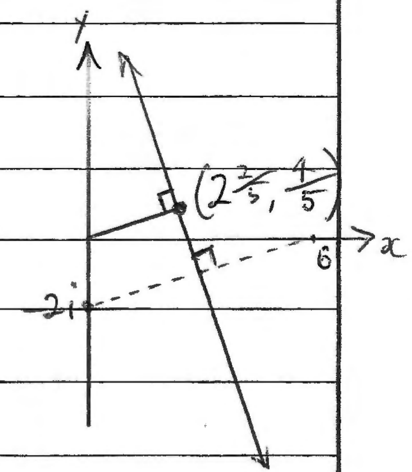
$|x-6+iy| = |x + i(y+2)|$

$\sqrt{(x-6)^2 + y^2} = \sqrt{x^2 + (y+2)^2}$ ①

$\cancel{x^2} - 12x + 36 + \cancel{y^2} = \cancel{x^2} + \cancel{y^2} + 4y + 4$

$0 = 12x + 4y - 32$

$0 = 3x + y - 8$ ②



(ii) Min. value of $|z|$ is $\perp$ distance of this line from $(0,0)$.

Need to find perpendicular line to P passing through origin.

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Gradient of z locus is -3

$\therefore$ Gradient of line through origin is $(0,0)$ is $\frac{1}{3}$. $\therefore$ Eq'n is $y = \frac{x}{3}$

Solve simultaneously:

$$3x + \frac{x}{3} - 8 = 0$$

$$\frac{10x}{3} = 8$$

$$10x = 24$$

$$x = 2\tfrac{2}{5} \quad \therefore y = \frac{2\frac{2}{5}}{3} = \frac{4}{5} \quad (1)$$

$$\therefore \min |z| \text{ is } \sqrt{\left(\tfrac{12}{5}\right)^2 + \left(\tfrac{4}{5}\right)^2}$$

$$= \frac{\sqrt{160}}{5} = \frac{4\sqrt{10}}{5} \quad (2)$$

(2.53)

b) $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10 = (x^2 - 2ax + (a^2+b^2)) \times (x^2 - 2ax + (a^2+4b^2))$ (1)

Equating coefficients of x^3

$$-4ax^3 = -4x^3 \quad \therefore a = 1 \quad (1)$$

Equate constants

$$10 = (a^2+b^2)(a^2+4b^2)$$

$$10 = (b^2+1)(4b^2+1)$$

$$0 = 4b^4 + 5b^2 - 9$$

$$0 = 4b^4 + 9b^2 - 4b^2 - 9$$

$$0 = b^2(4b^2+9) - (4b^2+9)$$

$$0 = (b^2-1)(4b^2+9)$$

$$\therefore b = \pm 1$$

+ | 5
× | -36
9, -4

$\therefore$ Roots/zeros are $1 \pm i$, $1 \pm 2i$ (3)

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c(i) $V = \sqrt{2x}\, e^{-x^2}, \quad x \geq 0 \cdot 1$

$\frac{dv}{dx} = \sqrt{2x} \times -2x \times e^{-x^2} + e^{-x^2} \times \sqrt{2} \times \frac{1}{2} x^{-\frac{1}{2}}$

$= -2\sqrt{2}\, x^{3/2} e^{-x^2} + \frac{\sqrt{2}\, e^{-x^2}}{2\sqrt{x}}$ ①

$a = v\frac{dv}{dx} = \sqrt{2x}\, e^{-x^2}\left[-2\sqrt{2}\, x^{3/2} e^{-x^2} + \frac{e^{-x^2}}{\sqrt{2x}}\right]$

$= -4x^2 e^{-2x^2} + e^{-2x^2}$ ②

$= e^{-2x^2}(1-4x^2)$ as req'd

(ii) When $x = 0 \cdot 1$, $a = v\frac{dv}{dx} > 0$. Particle increases speed until $a = 0$ ie: when $1 - 4x^2 = 0$

$4x^2 = 1$

$x^2 = \frac{1}{4}$

$x = 0 \cdot 5$ (since $x \geq 0 \cdot 1$) ①

Sub. this into V gives

$V = \sqrt{2 \times 0 \cdot 5} \times e^{-0 \cdot 5^2}$

$V = e^{-0 \cdot 25}$ is max. speed. ②

c(iii) $V = \frac{dx}{dt} = \sqrt{2x}\, e^{-x^2}$

$\therefore \int_1^2 \frac{e^{x^2}\, dx}{\sqrt{2x}} = \int dt$ ①

$T = \int_1^2 \frac{e^{x^2}\, dx}{\sqrt{2x}}$ ②

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(iv)

x	1	1.5	2
$f(x)$	$\frac{e}{\sqrt{2}}$	$\frac{e^{2.25}}{\sqrt{3}}$	$\frac{e^{4}}{2}$

$$\therefore T = \frac{h}{2}\left[y_0 + 2y_1 + y_2\right]$$

$$= \frac{0.5}{2}\left[\frac{e}{\sqrt{2}} + \frac{2e^{2.25}}{\sqrt{3}} + \frac{e^{4}}{2}\right] \quad (1)$$

$$= 10.04. \quad (2)$$

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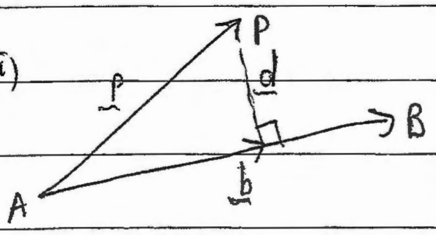
16.a) (i)

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} = \frac{-1\times(-2)+6\times(5)-1(-7)}{2^2+5^2+7^2}\left(-2\underset{\sim}{i}+5\underset{\sim}{j}-7\underset{\sim}{k}\right) \quad (1)$$

$$= \frac{1}{2}\left(-2\underset{\sim}{i}+5\underset{\sim}{j}-7\underset{\sim}{k}\right) \quad (2)$$

(ii)



$$d = \left|\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} - \underset{\sim}{p}\right| \quad (1)$$

$$= \left|\tfrac{1}{2}\left(-2i+5\underset{\sim}{j}-7\underset{\sim}{k}\right)-\left(-\underset{\sim}{i}+6\underset{\sim}{j}-\underset{\sim}{k}\right)\right|$$

$$= \left|0\underset{\sim}{i} - \tfrac{7}{2}\underset{\sim}{j} - \tfrac{5}{2}\underset{\sim}{k}\right|$$

$$= \frac{\sqrt{74}}{2} \quad (2)$$

or $\sqrt{\frac{37}{2}}$

b) (i) $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

In RHS, let $u = a - x$ when $x=0$, $u=a$

$du = -dx$ when $x=a$ $u=0$ (1)

$\therefore$ RHS becomes $\int_a^0 f(u) \times -du$

$$= \int_0^a f(u)\,du \quad (2)$$

= LHS (different variable irrelevant)

(ii) $f(x) + f(a-x) = f(a)$

$$\int_0^a f(x)\,dx + \int_0^a f(a-x)\,dx = \int_0^a f(a)\,dx$$

$$2\int_0^a f(x)\,dx = \left[f(a)\,x\right]_0^a \quad (1)$$

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$$2\int_0^a f(x)\,dx = af(a)$$

$$\therefore \int_0^a f(x)\,dx = \frac{a}{2}f(a) \quad ②$$

c)(i) $f(t) = \sin(a+nt)\sin b - \sin a\sin(b-nt)$

$= (\sin a\cos nt + \sin nt\cos a)\sin b -$

$\sin a(\sin b\cos nt - \sin nt\cos b)$ ①

$= \cos a\sin b\sin nt + \sin a\cos b\sin nt$

$= (\cos a\sin b + \sin a\cos b)\sin nt$

$= \sin(a+b)\sin nt$ ②

(ii) $\dfrac{\sin(a+nt)}{\sin(b-nt)} = \dfrac{\sin a}{\sin b}$

$\therefore \sin b\sin(a+nt) = \sin a\sin(b-nt)$

$\sin b\sin(a+nt) - \sin a\sin(b-nt) = 0$

$\Rightarrow \sin(a+b)\sin nt = 0$ using (i) ①

as $\sin(a+b) \neq 0$,

$\sin nt = 0$ ②

$nt = k\pi \quad k = 0, \pm1, \pm2 \ldots$

$t = \frac{k\pi}{n} \quad k = 0, \pm1, \pm2, \pm3 \ldots$

③

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d) Area of lower rectangle $= \frac{1}{n}$

Area of upper rectangle $= \frac{1}{n-1}$

$$\therefore \frac{1}{n} < \int_{n-1}^{n} \frac{dx}{x} < \frac{1}{n-1}$$

$$\frac{1}{n} < \left[\ln x\right]_{n-1}^{n} < \frac{1}{n-1}$$

$$\frac{1}{n} < \ln \frac{n}{n-1} < \frac{1}{n-1}$$

$$e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}} \quad ①$$

$$e < \left(\frac{n}{n-1}\right)^n < e^{\frac{n}{n-1}}$$

$$\frac{1}{e} > \left(\frac{n-1}{n}\right)^n > \frac{1}{e^{\frac{n}{n-1}}}$$

$$\frac{1}{e^{\frac{n}{n-1}}} < \left(\frac{n-1}{n}\right)^n < \frac{1}{e}$$

$$e^{\frac{-n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1} \quad \text{as req'd.}$$

②